

# $\Lambda$ -inflation and CMB anisotropy

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## Abstract

We explore a broad class of three-parameter inflationary models, called the  $\Lambda$ -inflation, and its observational predictions: high abundance of cosmic gravitational waves consistent with the Harrison-Zel'dovich spectrum of primordial cosmological perturbations, the non-power-law wing-like spectrum of matter density perturbations, high efficiency of these models to meet current observational tests, and others. We show that a parity contribution of the gravitational waves and adiabatic density perturbations into the large-scale temperature anisotropy,  $T/S \sim 1$ , is a common feature of  $\Lambda$ -inflation; the maximum values of  $T/S$  (basically not larger than 10) are reached in models where (i) the local spectrum shape of density perturbations is flat or slightly red ( $n_s \lesssim 1$ ), and (ii) the residual potential energy of the inflaton is near the GUT scale ( $V_0^{\frac{1}{4}} \sim 10^{16} \text{GeV}$ ). The conditions to find large  $T/S$  in the paradigm of cosmic inflation and the relationship of  $T/S$  to the ratio of the power spectra,  $r$ , and to the inflationary  $\gamma$  and Hubble parameters, are discussed. We argue that a simple estimate,  $T/S \simeq 3r \simeq 12\gamma \simeq \left(\frac{H}{6 \times 10^{13} \text{GeV}}\right)^2$ , is true for most known inflationary solutions and allows to relate straightforwardly the important parameters of observational and physical cosmology.

# 1 Introduction

The situation in physical cosmology is currently governed by experiment (observations) which made an increasing progress for the recent years. However, the theory of formation of *Large Scale Structure* in the Universe leaves something to be desired while a progress is still there: the simplest versions of the dynamical *Dark Matter* are discarded (e.g. sHDM, sCDM, cosmic strings), the cosmological model has become multiparametrer ( $\Omega_M, \Omega_\Lambda, \Omega_b, h, n_S, T/S$ , etc.) which hints on a complex nature of the dark matter in the Universe. Hopefully, the ongoing and oncoming measurements of the CMB anisotropy (both ground and space based) as well as the development of median and low  $z$  observations will fix the DM/LSS model of the Universe by a few per cent in the nearest future.

A theory of the very early Universe which meets most predictions and observational tests is inflation. It prophesys small Gaussian *Cosmological Density Perturbations* (the *Scalar* adiabatic mode) responsible for the LSS formation in the observable Universe. The ultimate goal here would be the reconstruction of DM parameters and CDP power spectrum directly from observational data (LSS *vs*  $\Delta T/T$ ).

A drama put in the basis of cosmic inflation is that it provides also a general ground for the fundamental production of *Cosmic Gravitational Waves* (the *Tensor* mode) which should contribute along with the S-mode into the  $\Delta T/T$  anisotropy at large angular scale<sup>1</sup>. Hence, a principal question on the way to the S-spectrum restoration remains the T/S problem – the fraction of the variance of the CMB temperature fluctuations on 10 degrees of arc generated by the CGWs:

$$(\Delta T/T)^2|_{10^\circ} = S + T. \quad (1)$$

Observational separation between the modes is postponed by the time when polarization measurements of the CMB anisotropy will be available (which

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<sup>1</sup> Obviously, all three modes of the perturbations of gravitational field – scalar, vector and tensor (see [1]) – induce the CMB anisotropy through the SW-effect [2]. However, most of the inflationary models considered by now are based on scalar inflaton fields which cannot be a source for the vector mode. A general physical reason for the production of the T and S perturbations in the expanding Universe is the *parametric amplification effect* [3], [4]: the spontaneous creation of quantum physical fields in a non-stationary gravitational background.

require the detector sensitivity  $\lesssim 1\mu\text{K}$ ). Today, we can investigate the T/S problem theoretically.

A common suggestion created by the *Chaotic Inflation* [5], that 'T/S is usually small ( $T/S \lesssim 0.1$ )', stems actually from a very specific property of the CI model (it inflates only at high values of the inflaton,  $\varphi > 1$ ). However, in general this is not true: any inflation produces inevitably *both* perturbation modes, the ratio between them is not limited by unity and sticks to the parameters of a given model<sup>2</sup>. Nevertheless, people often relate this T/S-CI feature to another basic property of the chaotic inflation with a smooth inflaton potential  $V(\varphi)$  – the *Harrison-Zel'dovich* S-spectrum ( $n_S \simeq 1$ ).

Such a mythological statement that 'T/S is small when  $n_S \simeq 1$ ', has even been strengthened by the power-law inflation [6], [7] which has displayed that T/S may become large only at the expense of the rejection from the HZ-spectrum in S-mode:  $T/S \gtrsim 1$  when  $n_S \leq 0.8$ ; obviously, *vice versa*, when  $n_S \rightarrow 1$ , the T/S tends to zero in a total accordance with the previous CI-assertion. The analytic approximation for T/S found in this model looks eventually universal for any inflationary dynamics when related to the T-spectrum slope index (estimated in the appropriate scale  $k_{COBE} \sim 10^{-3}h/\text{Mpc}$ )<sup>3</sup>,

$$\frac{T}{S} \simeq -6n_T \simeq 12\gamma. \quad (2)$$

Since a case for the *red* S-spectra suggested by power-law inflation ( $n_S < 1$ ) has confirmed the above statement of the T/S smallness for HZ-CDPs, we are facing to check the opposite situation – a case for models where the *blue* spectra ( $n_S > 1$ ) are allowed and the T/S there. An example of the blue S-spectrum is provided by (i) the two-field hybrid inflation [8], [9], [10], [11] for a certain range of the model parameters, (ii) a single massive inflaton [12] ( $V = V_0 + m^2\varphi^2/2$ ), and (iii) that producing power-law S-spectrum [13], [14], [15]. However, the problem of blue S-spectra is more generic and requires its full investigation. In this paper we present such analysis for the case of a

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<sup>2</sup>There is no fundamental theorem restricting T/S relative to the unity: the inflationary requirement,  $\gamma \equiv -\dot{H}/H^2 < 1$ , imposes only a wide constraint,  $T/S \lesssim 10$ , obviously insufficient to discriminate the T-mode in the cosmological context (see eqs.(2),(5)).

<sup>3</sup>It is just because the CGW spectrum created in *any* inflation is intrinsically akin to the evolution of the Hubble factor at the horizon crossing time:  $n_T \simeq -2\gamma < 0$ . Notice the T-spectrum stays always red in the minimally coupled gravity since a systematic decrease in time of the Hubble factor, cf.eq.(12).

single inflaton field.

Below, we start considering the inflationary requirements for the production of blue S-spectra. We introduce a simple natural model of such an inflation with one scalar  $\varphi$  field which we call the  $\Lambda$ -inflation. It proceeds at any values of the inflaton and generates a typical feature in the S-spectrum: a blue branch in short wavelengths (small  $\varphi$  values) and a red one in large wavelengths (high  $\varphi$  values). Between these two asymptotics the broad-scale transient spectrum region is settled down where the *'T/S is close to its highest value (generally not more than 10) when the S spectrum (or the joint S+T metric fluctuation spectrum) is essentially HZ one'*. Further on, we analyse physical reasons for the latter generic statement (CI is the measure zero in the family of  $\Lambda$ -inflation models) and its place in the inflationary paradigm. Surprisingly, the phenomena of large T/S and blue S-spectra are two totally disconnected problems: both are realized in  $\Lambda$ -inflation but at different scales and field values. The large T/S is produced where inflation proceeds only marginally (the subunity  $\gamma$ -values) which occurs near  $\varphi \sim 1$  where the S-spectrum tilt is slightly red,  $n_S \lesssim 1$ . On the contrary, the blueness ( $n_S > 1$ ) is gained for  $\varphi \ll 1$  and has thus a different physical origin. We conclude by discussing the necessary and sufficient conditions for obtaining large T/S from inflation, and argue for a general estimate of T/S based on eq.(2).

## 2 The $\Lambda$ -Inflation

We are looking for the simplest way to get a blue-kind spectrum of density perturbations generated at inflation driven by one scalar field  $\varphi$ .

The minimal coupling of  $\varphi$  to geometry is given by the action ( $c = \hbar = 8\pi G = 1$ ):

$$W[\varphi, g^{ik}] = \int \left( L - \frac{1}{2} R \right) \sqrt{-g} d^4x \quad (3)$$

where  $g_{ik}$  and  $R_{ik}$  are the metric and Ricci tensors respectively, with the signature  $(+ - - -)$ ,  $g = \det(g_{ik})$ , and  $R \equiv R^i_i$ . The field Lagrangian is an arbitrary function of two scalars,

$$L = L(w, \varphi), \quad (4)$$

where  $w^2 = \varphi_{,i} \varphi^{,i}$  is the kinetic term of  $\varphi$ -field.

Actually, the latter can be simplified at inflation. Indeed, the inflationary condition (taken in the locally flat Friedmann geometry),

$$\gamma \equiv -\frac{\dot{H}}{H^2} = \frac{3(\rho + p)}{2\rho} = \frac{3w^2M^2}{2(w^2M^2 - L)} < 1, \quad (5)$$

implies generally that

$$w^2M^2 \equiv \frac{\partial L}{\partial(\ln w)} < -2L,$$

just telling us on the validity of the Taylor-decomposition of (4) over small  $w^2$ :

$$L = L(0, \varphi) + \frac{1}{2}w^2M^2(0, \varphi) + 0(w^4),$$

where  $\rho \equiv w^2M^2 - L$  and  $p \equiv L$  are comoving density and pressure of  $\varphi$ -field,  $H = \frac{g_{,i}\varphi^{,i}}{6wg}$  is the local Hubble factor. After redefining the field by a new one,

$$\varphi \Rightarrow \int M(0, \varphi) d\varphi,$$

we come to a standard form for the Lagrangian density at inflation which is assumed further on:

$$L = -V(\varphi) + \frac{w^2}{2}. \quad (6)$$

Here  $V = V(\varphi)$  is the potential energy of  $\varphi$ -field.

A simple guess on the condition necessary to arrange inflation with a blue S-spectrum arises when we address an example of the slow-roll approximation. Under this approach the spectrum of created scalar perturbations  $q_k$  is straightforwardly related to the inflaton potential  $V(\varphi) \simeq 3H^2$  at the horizon-crossing:

$$q_k \simeq \frac{H}{2\pi\sqrt{2\gamma}} = \frac{H^2}{4\pi H'_\varphi}, \quad k = aH = \dot{a}, \quad (7)$$

where  $a$  is the scale factor and dot denotes the time derivative. The wave number  $k$  increases with time as  $a$  grows faster than  $H^{-1}$  in any inflationary expansion (see eq.(5)):

$$(\ln(aH))' = (1 - \gamma)H > 0.$$

Eq.(7) prompts evidently: while decreasing  $H'_\varphi$  with  $k > k_{cr}$ , one gains the power on short scales and, thus, realizes the blue spectrum slope.

Without loss of generality, we will assume that  $V(\varphi)$  is a function growing with  $\varphi(> 0)$  and getting its local minimum at  $\varphi = 0$ . It means that during the process of inflation  $\varphi$ -field evolves to smaller values. Hence, the necessary condition for a blue spectrum could be any way of flattenning the potential shape at smaller  $\varphi < \varphi_{cr}$  to provide for a rise of  $H^2/H'_\varphi$  and keeping the inflation still on ( $H'_\varphi < H/\sqrt{2}$ , cf. eqs.(5), (6)):

$$1 - n_S \simeq \frac{\gamma}{H} \left( \frac{H^2}{H'_\varphi} \right)'_\varphi < 0. \quad (8)$$

The latter equation leads to a broad-brush requirement of the positive potential energy at the local minimum point of  $\varphi$ -field:

$$V_0 \equiv V(0) > 0, \quad (9)$$

which displays the existence of the effective  $\Lambda$ -term during the period of inflation dominated by the residual (constant) potential energy:

$$V(\varphi < \varphi_{cr}) \simeq V_0 \equiv \Lambda \equiv 3H_0^2, \quad (10)$$

where the characteristic value  $\varphi_{cr}$  is determined as follows <sup>4</sup>:

$$V(\varphi_{cr}) = 2V_0. \quad (11)$$

This appearance of the *de Sitter*-type inflation (for  $\varphi < \varphi_{cr}$ ) results in a drastic difference with CI which has eventually assumed that  $V_0 = 0$  just making the inflation at small  $\varphi$  in principal impossible. Obviously, the latter hypothesis on vanishing the potential energy at  $\varphi = 0$  has reduced the CI-model to a very partial case (from the point of view of eq.(9)) restricting the inflation dynamics by only high values of the inflaton ( $\varphi > 1$ ). So, we may conclude that the  $\Lambda$ -inflation based on eq.(9) presents a general class of the fundamental inflationary models. In this sense they are more natural models (the CI being of the measure zero in  $V_0$ -parameter) allowing the inflation also at small  $\varphi$ -values (less than the Planckian one).

Summarising, we see that under condition (9) we have two qualitatively different stages of the inflationary dynamics separated by  $\varphi \sim \varphi_{cr}$ . We will call them:

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<sup>4</sup>In most applications  $\varphi_{cr} \sim 1$ , see eq.(32).

- the CI stage ( $\varphi \gtrsim \varphi_{cr}$ ), where the evolution is not influenced by the  $\Lambda$ -term and looks essentially like in standard chaotic inflation, and
- the dS stage ( $\varphi < \varphi_{cr}$ ) predominated by the  $V_0$ -constant.

The completion of the full inflation in this model is related to  $V_0$  -decay which is supposed to happen at some  $\varphi^* < \varphi_{cr}$  <sup>5</sup>. So, we deal with the three-parameter model  $(V_0, \varphi_{cr}, \varphi^*)$  starting as CI ( $\varphi > \varphi_{cr}$ ) and processing by dS-inflation at small  $\varphi$  ( $\varphi^* < \varphi < \varphi_{cr}$ ).

As we know from the CI theory smooth  $V$ -potentials create generally the *red*  $q_k$ -spectra ( $n_S < 1$  for  $\varphi > \varphi_{cr}$ ). On the other hand, eq.(9) provides physical grounds for the *blue* spectra generated at dS period ( $n_S > 1$ , cf. eq.(8)). Recall for comparison, that the spectrum of gravitational waves produced at any inflationary regime is given by the universal formula (here both polarizations are taken into account):

$$h_k = \frac{H}{\pi\sqrt{2}}, \quad k = aH, \quad (12)$$

which generally ensures the red-like T-spectra as  $H$  decreases in time for  $\rho + p > 0$ :  $n_T = -2\gamma < 0$  (see eq.(5)).

A trivial way to maintain eq.(9) is the introduction of an additive  $\Lambda$ -term in the inflation potential. Keeping in mind only the simplest dynamical terms we easily come to a trivial and rather general potential form:

$$V = V_0 + \frac{1}{2}m^2\varphi^2 + \frac{1}{4}\lambda\varphi^4, \quad (13)$$

which may also be understood as a decomposition of  $V(\varphi)$  over small  $\varphi$ . Here, such decomposition is a reasonable approach since the inflation proceeds to small  $\varphi \rightarrow 0$ . Obviously, eq.(11) can be explicitly reversed in this case:

$$\varphi_{cr}^2 = \frac{4V_0}{m^2 + \sqrt{m^4 + 4\lambda V_0}}. \quad (14)$$

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<sup>5</sup>We do not discuss here possible mechanisms for such metastability (it may be the coupling to other physical fields, a way of double- or platoo-like inflations, etc.) and take the  $\varphi^*$  value as an arbitrary parameter of our model (allowing to recalculate  $k_{cr}$  in Mpc). Mind that in CI  $\varphi^* \simeq \varphi_{cr}$ .

Also, we will use later the power-law potential

$$V = V_0 + \frac{\lambda_\kappa}{\kappa} \varphi^\kappa = V_0 (1 + y^\kappa), \quad (15)$$

where  $\kappa$  and  $\lambda_\kappa$  are positive numbers ( $\kappa \geq 2$ ,  $\lambda_2 \equiv m$ ,  $\lambda_4 \equiv \lambda$ ),  $\varphi_{cr}^\kappa = \kappa V_0 / \lambda_\kappa$ , and  $y = \varphi / \varphi_{cr}$ .

Let us turn to the evolution and spectral properties of  $\Lambda$ -inflation models.

### 3 The background model

Below, we consider dynamics under the condition (9).

The background geometry is classical employing the 6-parametric Friedmann group:

$$ds^2 = dt^2 - a^2 d\vec{x}^2 = a^2 (d\eta^2 - d\vec{x}^2), \quad (16)$$

The functions of time  $a$  and  $\varphi$  are found either from the Einstein equations:

$$H^2 = \frac{1}{3}V + \frac{1}{6}\dot{\varphi}^2, \quad (17)$$

$$\dot{H} = -\frac{1}{2}\dot{\varphi}^2, \quad (18)$$

or equivalently, from the  $\varphi$ -field equation (with  $H$  taken from eq.(17)):

$$\ddot{\varphi} + 3H\dot{\varphi} + V'_\varphi = 0. \quad (19)$$

Coming to the dimensionless quantities,

$$\begin{aligned} h &\equiv \frac{H}{H_0}, \quad v = v(y) \equiv \left(\frac{V}{V_0}\right)^{1/2}, \\ y &\equiv \frac{\varphi}{\varphi_{cr}}, \quad x \equiv H_0(t - t_{cr}), \quad \epsilon \equiv \frac{2}{\varphi_{cr}}, \end{aligned} \quad (20)$$

we can derive the first-order-equation for the function  $h = h(y)$ <sup>6</sup>:

$$h = \frac{v}{\sqrt{1 - \gamma/3}}, \quad \sqrt{2\gamma} = \epsilon \frac{h'}{h}, \quad (21)$$

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<sup>6</sup> Hereafter, the prime/dot will denote the derivative over  $y/x$ , i.e. the normalized  $\varphi/t$ , respectively.



and/or the second-order-equation for  $y = y(x)$ :

$$\ddot{y} + 3h\dot{y} + \frac{3}{2}\epsilon^2 vv' = 0. \quad (22)$$

Eq.(18) yields the relationship between two functions:

$$2\dot{y} = -\epsilon^2 h'. \quad (23)$$

The inflation condition (5) allows to find the inflationary solution of eq.(21) via the decomposition over small  $\gamma$ :

$$h = v \left( 1 + \frac{1}{6}\gamma + o(\gamma) \right), \quad (24)$$

$$a = -\frac{1}{H\eta} (1 + \gamma + o(\gamma)), \quad (25)$$

where

$$\sqrt{2\gamma} = \frac{\epsilon v'/v}{1 - \vartheta/3}, \quad \vartheta \equiv \frac{\epsilon \left( \sqrt{\gamma/2} \right)'}{1 - \gamma/3} = \frac{\left( \sqrt{2\gamma} \right)'_{\varphi}}{1 - \gamma/3}. \quad (26)$$

Making use of eqs.(23), (25) we may also present the derivatives of  $y$ -function over the conformal time,

$$\frac{dy}{d \ln |\eta|} = \epsilon \sqrt{\frac{\gamma}{2}} (1 + \gamma + o(\gamma)), \quad \vartheta = \frac{d \ln \sqrt{\gamma}}{d \ln |\eta|} \left( 1 - \frac{2}{3}\gamma + o(\gamma) \right). \quad (27)$$

We will also need for further analysis the  $\varphi$ -derivatives at the horizon-crossing<sup>7</sup>,

$$\frac{d\varphi}{d \ln k} = -\sqrt{2\gamma} (1 + \gamma + o(\gamma)), \quad \frac{d \ln \gamma}{d \ln k} = -2\vartheta \left( 1 + \frac{2}{3}\gamma + o(\gamma) \right), \quad (28)$$

and the scattering potentials (cf.eq.(52)),

$$U \equiv \frac{d^2 \left( a\sqrt{\gamma} \right)}{a\sqrt{\gamma} d\eta^2} = a^2 H^2 \left( 2 - \gamma - 3\vartheta \left( 1 - \frac{\gamma}{3} \right)^2 + \frac{1}{4}\epsilon^2 \gamma'' \right),$$

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<sup>7</sup>Eq.(18) yields

$$\frac{d\varphi}{d \ln a} = -\sqrt{2\gamma}, \quad \frac{d^2 \varphi}{d(\ln a)^2} = \frac{d\gamma}{d\varphi},$$

the (-) sigh implies that  $\varphi$  decreases with time.

$$U^\lambda \equiv \frac{d^2 a}{a d\eta^2} = a^2 H^2 (2 - \gamma) = \frac{2}{\eta^2} \left( 1 + \frac{3}{2} \gamma + o(\gamma) \right). \quad (29)$$

Actually, eqs.(24)-(29) are true during the whole period of inflation based on inequality (5); they describe the evolution along the attractor inflationary separatix towards which any solution of eqs.(17)-(19) tends during the Universe expansion.

However, it is an additional to the inflation condition (5) assumption known as the slow-roll approximaion,

$$|\vartheta| < 1, \quad (30)$$

that, when works, simplifys the situation allowing to relate  $\gamma$  and  $y$  algebraically (see eqs.(26)) and thus to solve eqs.(21), (26) explicitly. Both inequalities (5) and (30) can be rewritten, respectively, as

$$\epsilon \frac{v'}{v} < 1, \quad \text{and} \quad \epsilon^2 \frac{|v''|}{v} < 1. \quad (31)$$

$\Lambda$ -inflation proceeds most difficult near  $y \sim 1$ . Indeed, for the power-law potential (15),  $v = \sqrt{1 + y^\kappa}$ , the first inequality (31) meets at the worst point  $y \sim y_1 = (\kappa - 1)^{\frac{1}{\kappa}} \simeq 1$  only for small  $\epsilon$ ,

$$\epsilon < \epsilon_0 = \frac{2}{\kappa - 1}, \quad \text{or} \quad \varphi_{cr} \gtrsim (\kappa - 1) \geq 1, \quad (32)$$

that we assume hereafter. The second inequality (31) holds at any  $y$  unless  $\kappa < 3$ . For the latter case the slow-roll approximation is broken in the field interval

$$\exp \left( -\frac{1}{\kappa - 2} \right) < y < 1, \quad (33)$$

where the left-hand-side keeps constant:  $\frac{\epsilon^2 v''}{v} \sim \epsilon^2$  (hence, the slow-roll approximation is restored in the limit  $\epsilon \rightarrow 0$ ).

So, for  $\kappa = 2$ , the whole evolution for  $y < 1$  deviates strongly from the slow-roll approximation. Before coming to it, we write down the evolution for  $\kappa \geq 3$ .

### 3.1 The $\Lambda\lambda$ -Inflation

The slow-roll approximation is met for  $\kappa \geq 3$ ; then, under conditions (5) and (30), eq.(23) is integrated explicitly:

$$a = \exp \left( - \int \frac{d\varphi}{\sqrt{2\gamma}} \right) \simeq \gamma^{\frac{1}{6}} \exp \left( - \frac{2}{\epsilon^2} \int \frac{v dy}{v'} \right). \quad (34)$$

Substituting here  $v = \sqrt{1 + y^\kappa}$ , we have at the horizon-crossing:

$$\kappa \geq 3 : \quad y^2 \left( 1 - \left( \frac{y_2}{y} \right)^\kappa \right) = \Theta, \quad (35)$$

where  $y_2 = \left( \frac{2}{\kappa-2} \right)^{\frac{1}{\kappa}} \simeq 1$ ,  $\Theta = -\frac{\kappa\epsilon^2}{2} \ln K = \frac{\kappa-4}{\kappa-2} - \frac{\kappa\epsilon^2}{2} \ln K_c$ ,  $K = \frac{a}{\gamma^{\frac{1}{6}}} = \left( \frac{k}{k_2} \right) \left( \frac{y_2}{y} \right)^{\frac{\kappa-1}{3}} \left( \frac{\kappa/(\kappa-2)}{1+y^\kappa} \right)^{\frac{1}{6}} \sim \frac{k}{k_2}$ ,  $K_c = \left( \frac{k}{k_{cr}} \right) \left( \frac{1}{y} \right)^{\frac{\kappa-1}{3}} \left( \frac{2}{1+y^\kappa} \right)^{\frac{1}{6}} \sim \frac{k}{k_{cr}}$ . Evidently,

$$\frac{d \ln K_{(c)}}{d \ln k} = 1 + \gamma + \vartheta/3 + o(\gamma) + o(\vartheta) \simeq 1,$$

$$y \simeq \begin{cases} \Theta^{\frac{1}{2}}, & y > y_2, \\ \left( \frac{2}{(\kappa-2)|\Theta|} \right)^{\frac{1}{\kappa-2}}, & y < y_2. \end{cases}$$

The transition period between these two asymptotics,  $|\Theta| \lesssim 1$ , is pretty small in  $y$ -space,

$$|y - y_2| < \frac{1}{\kappa} : \quad y \simeq y_2 + \frac{1}{\kappa y_2} \Theta \simeq 1 - \frac{\epsilon^2}{2} \ln K,$$

however, it is big in the correspondent frequency band (cf.eq.(32)):

$$|\ln K| < \frac{2}{\kappa\epsilon^2} \left( \gtrsim \frac{1}{\epsilon} \right). \quad (36)$$

An interesting physical case here is the case with self-interacting field, which we call  $\Lambda\lambda$ -inflation:

$$\kappa = 4 : \quad y^2 \simeq \sqrt{1 + (\epsilon^2 \ln K)^2} - \epsilon^2 \ln K, \quad (37)$$

where  $K = K_c = \frac{k}{k_{cr} y} \left( \frac{2}{1+y^4} \right)^{\frac{1}{6}}$ . Recall that the  $\epsilon$ -parameter should not exceed unity if we want to keep inflation everywhere.

### 3.2 The $\Lambda m$ -Inflation

The case of massive field ( $\kappa = 2$ ,  $v = \sqrt{1 + y^2}$ ) violates the slow-roll condition and requires more careful investigation.

The slow-roll approximation works well for  $y \gtrsim 1$ , but is broken at small  $y$  as  $\frac{v''}{v} = v^{-4} \sim 1$  for  $y < 1$  (see eq.(33)). In the latter case  $h \simeq 1$  and eq.(22) turns to linear one presenting the  $y$ -function as a linear superposition of the *fast* (+) and *slow* (-) exponents ( $\sim e^{-1.5(1 \pm p)x} \sim |\eta|^{1.5(1 \pm p)}$ ). This allows for a straightforward, i.e. independent of the exponent amplitudes, derivation of the  $U$ -potential at the dS stage (see eqs.(27),(29)):

$$y < 1: \quad U \equiv \frac{d^2(a\sqrt{\gamma})}{a\sqrt{\gamma}d\eta^2} \simeq \frac{d^3 y}{d\eta^3} \left( \frac{dy}{d\eta} \right)^{-1} \simeq \frac{9p^2 - 1}{4\eta^2}, \quad (38)$$

where  $p = \sqrt{1 - \frac{2\epsilon^2}{3}}$ .

The inflationary evolution proceeds in a non-oscillatory way for  $\varphi < \varphi_{cr}$  if

$$0 < p < 1, \quad \varphi_{cr} \gtrsim 1.6, \quad (39)$$

that we will assume further on. With such a requirement the inflation is guaranteed for any  $\varphi$  (cf. eqs.(32)).

To find the exponent amplitudes for  $y(< 1)$  we have to match the full inflationary separatrix at  $y \sim 1$ . To do it let us exclude the first-derivative term in eq.(22) introducing a new variable  $z = z(\eta) \equiv ya$ :

$$\frac{d^2 z}{d^2 \eta^2} - \tilde{U}z = 0, \quad (40)$$

and then approximate the  $\tilde{U}$ -function by a simple step-function:

$$\tilde{U} \equiv (aH)^2 \left( 2 - \gamma - \frac{3\epsilon^2}{2h^2} \right) \simeq \frac{2}{\eta^2} \left( 1 - \frac{3\epsilon^2}{4v^2} \right) \simeq \frac{1}{\eta^2} \begin{cases} 2, & \eta < \eta_3, \\ \frac{9p^2 - 1}{4}, & \eta > \eta_3, \end{cases}$$

where  $\eta_3 \simeq \eta_{cr}$ . The solution of eq.(40) is then got explicitly; matching  $z$ -function and its first derivative at  $\eta = \eta_3$  and taking into account that  $H_0 z \eta \rightarrow -1$  for large  $y$ , we obtain at the dS stage (cf. eqs.(27)):

$$\omega > 1: \quad y \simeq \omega^{-\frac{3}{2}} \left( \text{ch}\mu + \frac{1}{p} \text{sh}\mu \right), \quad \sqrt{\frac{\gamma}{2}} \simeq \frac{\epsilon}{p} \omega^{-\frac{3}{2}} \text{sh}\mu, \quad \vartheta \simeq \frac{3}{2} (1 - p \text{cth}\mu), \quad (41)$$

where  $\mu = \frac{3}{2}p \ln \omega_{\sim}^{\geq p}$ ,  $\omega = \frac{\eta_3}{\eta} \simeq \frac{\eta_{cr}}{\eta} \simeq \frac{k}{k_{cr}}$ .

The fitting coefficients in eq.(41) describe a part ( $y < 1$ ) of the full inflationary separatrix extending from large to small values of the  $\varphi$ -field<sup>8</sup>. We see that at the de Sitter stage the function  $\vartheta = \vartheta(\omega) > 0$  varies slowly,

$$y < 1 : \quad \vartheta \simeq \begin{cases} \frac{3}{2} - \frac{1}{\ln \omega}, & 1 \lesssim \ln \omega < \frac{2}{3p}, \\ \frac{\epsilon^2}{1+p}, & \ln \omega \gtrsim \frac{2}{3p} \end{cases}, \quad (42)$$

and

$$\sqrt{2\gamma} = \frac{\epsilon y}{1 - \frac{\vartheta}{3}}, \quad y \simeq \begin{cases} \frac{3}{2} \omega^{-\frac{3}{2}} \ln \omega, & 1 \lesssim \ln \omega < \frac{2}{3p}, \\ \frac{1+p}{2p} \omega^{\frac{3}{2}(p-1)}, & \ln \omega \gtrsim \frac{2}{3p} \end{cases}. \quad (43)$$

The field evolution approaches the slow exponent only for  $\ln \omega > \frac{2}{3p}$   $\left( y < \frac{\exp(-\frac{1}{p})}{p} \right)$ :

$$y \ll 1 : \quad y \simeq \frac{1+p}{2p} \omega^{\frac{3}{2}(p-1)}, \quad \sqrt{2\gamma} \simeq \frac{\epsilon}{p} \omega^{\frac{3}{2}(p-1)}. \quad (44)$$

For  $p \in (\frac{2}{3}, 1)$  the true evolution at the dS stage is presented only by the bottom lines in eqs.(42), (43); this fact is used in the Appendix to restore the whole inflation dynamics for  $\epsilon^2 < \frac{5}{6}$ .

Comparing eqs.(35) and (44) we see that at the dS stage  $y$  decays as  $\ln k$  for  $\kappa \geq 3$ , whereas it is the power-law for  $\kappa = 2$ . For the intermediate case  $2 < \kappa < 3$  the slow-roll approximation is violated only within the limited interval (33) where the solution can be matched by eq.(41) with  $p = \sqrt{1 - \frac{\kappa \epsilon^2}{3}}$ .

## 4 The generation of primordial perturbations

Below, we introduce the S and T metric perturbation spectra and find them for  $\Lambda$ -inflation.

The linear perturbations over the geometry (16) can be irreducibly represented in terms of the uncoupled Scalar, Vector and Tensor parts [1]. The vector perturbations are not induced in our case as scalar fields are not their

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<sup>8</sup> The fitting accuracy is quite satisfactory. Say, in the slow-roll approximation  $p \rightarrow 1$ :  $\frac{\eta_3}{\eta_{cr}} = 2^{-\frac{1}{6}} \simeq 1$  and  $y \omega^{1.5(1-p)} = \sqrt{e} \sim 1$ . See the Appendix for more detail.

sources. Under the action (3) we are rest with only the S and T modes, and the new geometry looks as follows:

$$ds^2 = (1 + h_{00}) dt^2 + 2ah_{0\alpha} dt dx^\alpha - a^2(\delta_{\alpha\beta} + h_{\alpha\beta}) dx^\alpha dx^\beta, \quad (45)$$

$$\frac{1}{2}h_{\alpha\beta} = A\delta_{\alpha\beta} + B_{,\alpha\beta} + G_{\alpha\beta}, \quad h_{0\alpha} = C_{,\alpha},$$

where  $G_\alpha^\alpha = G_{\alpha,\beta}^\beta = 0$ . The gravitational potentials  $h_{00}$ ,  $A$ ,  $B$ ,  $C$  are coupled to the perturbation of scalar field  $\delta\varphi$ , whereas  $G_{\alpha\beta}$  is the free tensor field. The Lagrangian  $L^{(2)}$  of the perturbation sector of the geometry (45) is given by decomposing the integrand (3) up to the second order in the perturbation amplitudes. Our further analysis of the S-sector follows a general theory of the  $q$ -field ([4], [16]), the gravitational waves are totally described by the gauge-independent 3D-tensor  $G_{\alpha\beta}$  ([3], [17], [18]).

Instead of considering gauge-dependent potentials ( $h_{00}$ ,  $A$ ,  $B$ ,  $C$ ,  $\delta\varphi$ ) we introduce the gauge-invariant canonical 4D-scalar  $q$  uniquely fixed by the appearance of the S-part of the perturbative Lagrangian  $L^{(2)}$  similar to a massless field:

$$L^{(2)} = L(q, G_{\alpha\beta}) = \frac{1}{2}\alpha^2 q_{,i} q^{,i} + \frac{1}{2}G_{\alpha\beta,\gamma} G^{\alpha\beta,\gamma}, \quad (46)$$

where  $\alpha^2 \equiv 2\gamma = \frac{\rho+p}{H^2} = \left(\frac{\dot{\varphi}}{H}\right)^2$ ,  $\alpha = \frac{\dot{\varphi}}{H}$  (mind the choice of the sign for  $\alpha$  that we take coinciding with the sign of  $\dot{\varphi}$ ). The relation of  $q$  to the original potentials takes the following form:

$$\begin{aligned} \delta\varphi &= \alpha(q + A), \quad a^2\dot{B} + C = \frac{\Phi + A}{H}, \\ \frac{1}{2}h_{00} &= \gamma q + \left(\frac{A}{H}\right), \quad \Phi = \frac{H}{a} \int a\gamma q dt, \\ \frac{\delta\rho}{\rho+p} &= \frac{\dot{q}}{H} - 3(q + A), \quad 4\pi G\delta\rho_c \equiv \gamma H\dot{q} = a^{-2}\Delta\Phi, \end{aligned} \quad (47)$$

where  $a$ ,  $\varphi$ ,  $H$ ,  $\alpha$ ,  $\gamma$ ,  $\rho = \frac{1}{2}w^2 + V$  and  $p = \frac{1}{2}w^2 - V$  are the background functions of time,  $\Phi$  is the "Newtonian" gauge-invariant gravitational potential related non-locally to  $q$ ,  $\Delta \equiv \partial^2/\partial^2\vec{x}^2$  is spatial Laplacian, ( $\Delta = -k^2$  in the Fourier representation,  $\delta\rho_c$  is the comoving density perturbation). Any two potential taken from the triple  $A$ ,  $B$ ,  $C$  are arbitrary functions of

all coordinates, which determines the gauge choice. All information on the physical scalar perturbations is contained in the  $q = q(t, \vec{x})$  field, the dynamical 4D-scalar propagating in the unperturbed Friedmann geometry (i.e. independently of any gauge in eq.(45)).

The equations of motions of the  $q$  and  $G_{\alpha\beta}$  fields are two harmonic oscillators:

$$\ddot{q} + \left(3H + \frac{\dot{\gamma}}{\gamma}\right) \dot{q} - a^{-2} \Delta q = 0, \quad (48)$$

$$G_{\alpha\beta}'' + 3H G_{\alpha\beta}' - a^{-2} \Delta G_{\alpha\beta} = 0. \quad (49)$$

A standard procedure to find the amplitudes generated is to perform the secondary quantization of the field operators,

$$q = \int_{-\infty}^{\infty} d^3 \vec{k} \left( a_{\vec{k}} q_{\vec{k}} + a_{\vec{k}}^+ q_{\vec{k}}^* \right), \quad (50)$$

$$G_{\alpha\beta} = \sum_{\lambda} \int_{-\infty}^{\infty} d^3 \vec{k} \left( a_{\vec{k}}^{\lambda} h_{\vec{k}\alpha\beta}^{\lambda} + a_{\vec{k}}^{\lambda+} h_{\vec{k}\alpha\beta}^{\lambda*} \right),$$

where  $+/*$  denotes Hermit/complex conjugation, index  $\lambda = +, \times$  runs two polarizations of gravitational waves with the polarization tensors  $c_{\alpha\beta}^{\lambda}(\vec{k})$ , and

$$q_{\vec{k}} = \frac{\nu_{\vec{k}}}{(2\pi)^{\frac{3}{2}} \alpha a} e^{i\vec{k}\vec{x}}, \quad (51)$$

$$h_{\vec{k}\alpha\beta}^{\lambda} = \frac{\nu_{\vec{k}}^{\lambda}}{(2\pi)^{\frac{3}{2}} a} e^{i\vec{k}\vec{x}} c_{\alpha\beta}^{\lambda}(\vec{k}),$$

$$\delta^{\alpha\beta} c_{\alpha\beta}^{\lambda}(\vec{k}) = k^{\alpha} c_{\alpha\beta}^{\lambda}(\vec{k}) = 0, \quad c_{\alpha\beta}^{\lambda}(\vec{k}) c^{\alpha\beta\lambda'}(\vec{k})^* = \delta_{\lambda\lambda'}.$$

The time-dependent  $\nu$ -functions satisfy the respective Klein-Gordon equations,

$$\frac{d^2 \nu_k^{(\lambda)}}{d\eta^2} + \left(k^2 - U^{(\lambda)}\right) \nu_k^{(\lambda)} = 0, \quad (52)$$

with  $U = U(\eta) \equiv \frac{d^2(\alpha a)}{\alpha a d\eta^2}$  for the  $q$ -field and  $U^{\lambda} = U^{\lambda}(\eta) \equiv \frac{d^2 a}{a d\eta^2}$  for each polarization of the gravitational waves  $\nu_k^{\lambda}$ . The standard commutation relations between the annihilation and creation operators,

$$[a_{\vec{k}} a_{\vec{k}'}^+] = \delta(\vec{k} - \vec{k}'), \quad [a_{\vec{k}}^{\lambda} a_{\vec{k}'}^{\lambda'+}] = \delta(\vec{k} - \vec{k}') \delta_{\lambda\lambda'},$$

require the following normalization condition for each of the  $\nu$ -functions:

$$\nu_k^{(\lambda)} \frac{d\nu_k^{(\lambda)*}}{d\eta} - \nu_k^{(\lambda)*} \frac{d\nu_k^{(\lambda)}}{d\eta} = i.$$

Eqs.(46)-(52) specify the *parametric amplification effect*: the production of the perturbations – the phonons for  $S$ -mode [4] and the gravitons for  $T$ -mode [3] – in the process of the Universe expansion (the latter is imprinted in the non-zero scattering potentials  $U^{(\lambda)}$  in eqs.(52)).

From the inflationary condition (5) one finds always  $k\eta \rightarrow -\infty$  for the early inflation (scales inside the horizon); therefore, the microscopic vacua states of the  $q$  and  $G_{\alpha\beta}$  fields mean the positive frequency choice for the initial  $\nu$ -functions:

$$k|\eta| \gg 1 : \quad \nu_k^{(\lambda)} = \frac{\exp(-ik\eta)}{\sqrt{2k}}. \quad (53)$$

So, the problem of the spontaneous creation of density perturbations and gravitational waves is finally reduced to solving the eqs.(52), (53) with the effective potentials  $U^{(\lambda)}$  taken from the inflationary background regimes considered above.

For the late inflation  $k\eta \rightarrow 0$  (scales outside the horizon), the perturbations become semiclassical since the fields are getting frozen in time and thus acquire the phase (only the *growing* solutions of eqs.(48),(49) survive in time)<sup>9</sup>,

$$k|\eta| \ll 1 : \quad q = q(\vec{x}), \quad G_{\alpha\beta} = G_{\alpha\beta}(\vec{x}). \quad (54)$$

One can, therefore, treat these time-independent perturbation fields as realizations of the classical random Gaussian fields with the following power spectra:

$$\langle q^2 \rangle = \int_0^\infty q_k^2 \frac{dk}{k}, \quad \langle G_{\alpha\beta} G^{\alpha\beta} \rangle = \int_0^\infty h_k^2 \frac{dk}{k},$$

---

<sup>9</sup> Here, the transfer from the quantum (squeezed) to classical case occurs when one neglects the *decaying* solutions of eqs.(48),(49) for  $\eta \rightarrow 0$ :

$$k|\eta| < 1 : \quad q_d \sim \int^0 \frac{d\eta}{a^2\gamma} = \frac{1}{3\gamma} H^2 \eta^3 (1 + O(\gamma)), \quad G_d \sim \int^0 \frac{d\eta}{a^2} = \frac{1}{3} H^2 \eta^3 (1 + O(\gamma)),$$

and thus is left only with the growing ones (see eq.(54)). This procedure turns the annihilation and creation operators into the  $C$ -numbers (where the commutators vanish).



$$q_k = \frac{k^{\frac{3}{2}} |\nu_k|}{2\pi a \sqrt{\gamma}}, \quad h_k = \frac{k^{\frac{3}{2}} \sqrt{|\nu_k^+|^2 + |\nu_k^\times|^2}}{\pi a \sqrt{2}} = \frac{k^{\frac{3}{2}} |\nu_k^\lambda|}{\pi a} \quad (55)$$

Here the  $\nu$ -functions are taken in the limit  $|\eta| \ll k^{-1}$ , and the gravitational wave spectra in both polarizations are identical. The local slopes and the ratio of the power spectra are found as follows:

$$n_S - 1 \equiv 2 \frac{d \ln q_k}{d \ln k}, \quad n_T \equiv 2 \frac{d \ln h_k}{d \ln k}, \quad r \equiv \left( \frac{h_k}{q_k} \right)^2 = 4 \left( \gamma \left| \frac{\nu_k^\lambda}{\nu_k} \right|^2 \right)_{k|\eta| \ll 1}. \quad (56)$$

Note, that the quantities  $q_k$ ,  $h_k$ ,  $n_S$ ,  $n_T$ ,  $r$  are the functions of the wavenumber only. For references, we recall also the density perturbation and Newtonian potential linked to the  $q$ -field in the Friedmann Universe (cf. eqs.(47), (54)),

$$k < aH : \quad \Delta_k = \frac{2}{3} \left( \frac{k}{aH} \right)^2 \Phi_k, \quad \Phi_k = \Gamma q_k,$$

where  $\Delta_k$ ,  $\Phi_k$  are the dimensionless spectra, respectively,

$$\left\langle \left( \frac{\delta \rho_c}{\rho} \right)^2 \right\rangle = \int_0^\infty \Delta_k^2 \frac{dk}{k}, \quad \langle \Phi^2 \rangle = \int_0^\infty \Phi_k^2 \frac{dk}{k},$$

and  $\Gamma = \frac{H}{a} \int a \gamma dt = 1 - \frac{H}{a} \int a dt$  is the function of time ( $\Gamma = (1 + \beta)^{-1} = \text{const}$  for the power-law expansion,  $a \sim t^\beta$ ).

## 5 The power spectra

When it works, the slow-roll approximation allows for simple derivation of the S-spectrum ( $U^{(\lambda)} \simeq 2/\eta^2$ , cf. eqs.(29)):

$$q_k \simeq \frac{H}{2\pi \sqrt{2\gamma}}, \quad h_k = \frac{H}{\pi \sqrt{2}}, \quad k = aH, \quad (57)$$

where  $H = H_0 v$ ,  $\sqrt{2\gamma} \simeq \epsilon \frac{v'}{v}$ ,  $\vartheta \simeq \frac{1}{2} \epsilon^2 \left( \frac{v'}{v} \right)'$ . The spectra ratio and the local slopes are then the following (see eqs.(28), (32), (56)):

$$r \simeq -2n_T = 4\gamma \simeq \frac{1}{2} \left( \frac{\epsilon \kappa y^{\kappa-1}}{1 + y^\kappa} \right)^2 \leq r_{max} = \frac{1}{2} \left( \frac{\epsilon (\kappa - 1)}{y_1} \right)^2 \simeq 2 \left( \frac{\epsilon}{\epsilon_0} \right)^2, \quad (58)$$

$$n_S - 1 \simeq 2(\vartheta - \gamma) = f(y), \quad f_- \leq f(y) \leq f_+,$$

where  $f(y) = \frac{\kappa}{2} y^{\kappa-2} \left( \frac{\epsilon}{1+y^\kappa} \right)^2 \left( \kappa - 1 - \frac{\kappa+2}{2} y^\kappa \right)$ ,  $y_\pm = \left( \kappa - 1 \mp \kappa \sqrt{\frac{\kappa-1}{\kappa+2}} \right)^{\frac{1}{\kappa}}$ ,  $f_\pm = f(y_\pm) = \frac{(\kappa-1)(\kappa+2)}{12} \left( \frac{\epsilon}{y_\pm} \right)^2 \left( \pm 2 \sqrt{\frac{\kappa-1}{\kappa+2}} - 1 \right) \simeq \pm \left( \frac{\epsilon}{\epsilon_0} \right)^2$ .

Eqs.(58) are true for  $v = \sqrt{1+y^\kappa}$ ; the T-spectrum deviates maximally from HZ (and the spectrum ratio reaches its maximum) at  $y_1 \simeq (\kappa - 1)^{\frac{1}{\kappa}} \simeq 1$ ; the S-spectrum achieves its minimum and becomes exactly HZ one at  $y_4 = \left( \frac{\kappa-1}{1+\frac{\kappa}{2}} \right)^{\frac{1}{\kappa}} = y_1 \left( \frac{2}{\kappa+2} \right)^{\frac{1}{\kappa}} \simeq 1$ , it is the most red (blue) at  $y_-$  ( $y_+$ ); the points  $y_1$  and  $y_4$  lay always inside the interval  $(y_+, y_-)$  while the region (36) resides there only if  $\kappa \leq 8$ . Eq.  $f(y) = \text{const} \in [f_-, f_+]$  has two solutions: one is located within the interval  $[y_+, y_-]$  where  $r$  is large,  $\frac{r}{r_{max}} > \left( \frac{\kappa+1}{3\kappa} \right)^2$  and  $r(n_S = 1) \simeq r_{max}$ ; another is outside this interval where  $r$  is small,  $\frac{r}{r_{max}} < 1$  and  $r(n_S = 1) = 0$ .

So, for  $\kappa \geq 3$  we have from eq.(35) the following asymptotics for the power spectra:

$$q_k^2 \simeq \left( \frac{H_0}{\epsilon \pi \kappa} \right)^2 \frac{(1+y^\kappa)^3}{y^{2\kappa-2}} \simeq \frac{\lambda_\kappa}{12\pi^2} \begin{cases} \kappa^{\frac{\kappa-4}{2}} |2 \ln K|^{\frac{\kappa+2}{2}}, & K < \exp \left( -\frac{2}{\kappa \epsilon^2} \right) \\ \left( \frac{V_0}{\lambda_\kappa} \right)^{\frac{\kappa-4}{\kappa-2}} ((\kappa-2) \ln K)^{\frac{2\kappa-2}{\kappa-2}}, & K > \exp \left( \frac{2}{\kappa \epsilon^2} \right) \end{cases},$$

$$h_k^2 = \frac{H_0^2}{2\pi^2} (1+y^\kappa) = \frac{1}{6\pi^2} \begin{cases} \frac{\lambda_\kappa}{\kappa} |2\kappa \ln K|^{\frac{\kappa}{2}}, & K < \exp \left( -\frac{2}{\kappa \epsilon^2} \right) \\ V_0, & K > \exp \left( \frac{2}{\kappa \epsilon^2} \right) \end{cases}.$$

In the transition region (36) the ratio of the spectra is approximately constant independent of the  $\kappa$ -index:  $r \simeq 2\epsilon^2$  (it is a factor  $\epsilon_0^{-2}$  less than  $r_{max}$ ).

For  $\Lambda\lambda$ -inflation the spectra are resolved explicitly (see eq.(37)):

$$\kappa = 4 : \quad \begin{aligned} q_k &\simeq \frac{1}{\pi} \sqrt{\frac{2\lambda}{3}} \left( \epsilon^{-4} + \ln^2 K \right)^{\frac{3}{4}}, \\ h_k &= \frac{H_0}{\pi} \left( 1 + \frac{\ln K}{\sqrt{\epsilon^{-4} + \ln^2 K}} \right)^{-\frac{1}{2}}, \end{aligned} \quad (59)$$

and  $y_1 = 3^{\frac{1}{4}}$ ,  $K_1 = \exp \left( -\frac{1}{\sqrt{3}\epsilon^2} \right)$ ,  $y_2 = y_4 = 1$ ,  $y_- = \frac{1}{y_+} = \left( \sqrt{2} + 1 \right)^{\frac{1}{2}}$ ,  $K_\pm = \exp \left( \mp \frac{1}{\epsilon^2} \right)$ . An example of the power spectra for  $\epsilon = 0.3$  is shown in Fig.1. Fig.2 clarifies the relation between  $r$  and  $n_S - 1$  for any  $\epsilon < 1$ . We see there is no correlation between the blueness and large  $r$ : the region of large  $r$ -values is located in the red and HZ sectors of the S-spectrum.

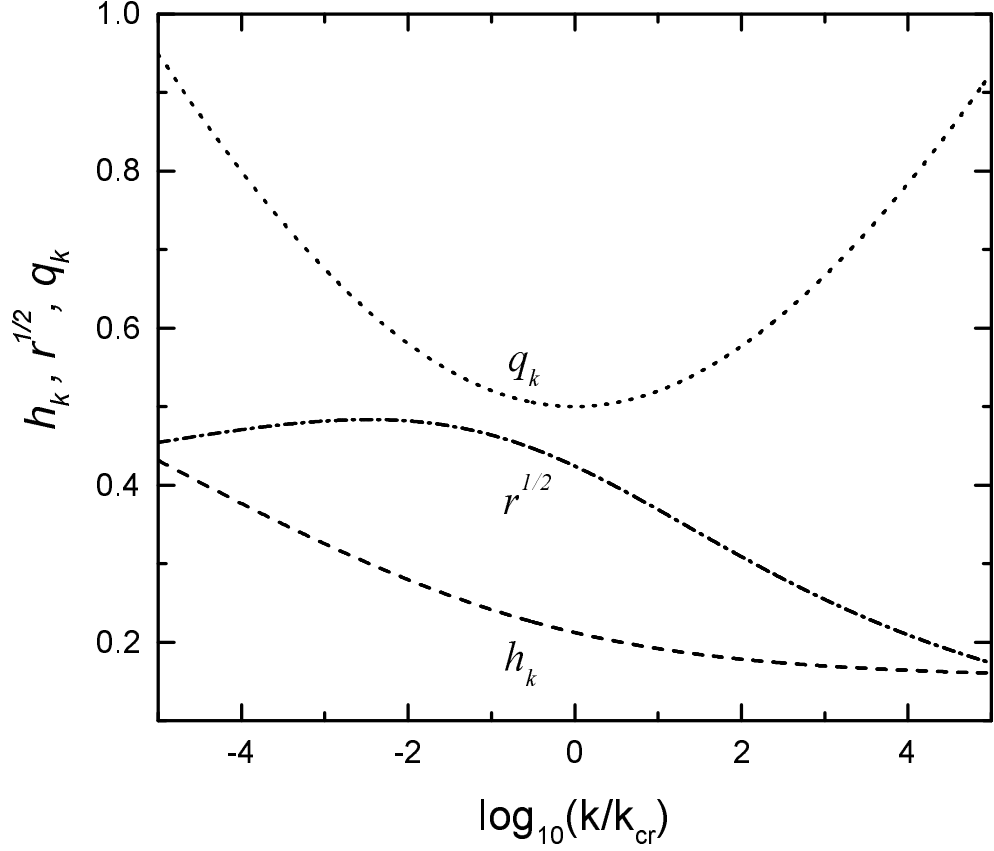


Figure 1: The spectra of scalar perturbations  $q_k$  (dotted curve), tensor perturbations  $h_k$  (dashed curve), and the ratio between them  $r^{\frac{1}{2}} \equiv h_k/q_k$  (dot-dashed curve), in the  $\Lambda\lambda$ -inflation model with  $\epsilon = 0.3$ . The normalization is arbitrary, however the ratio does not depend on normalization and is true for the used model parameter.

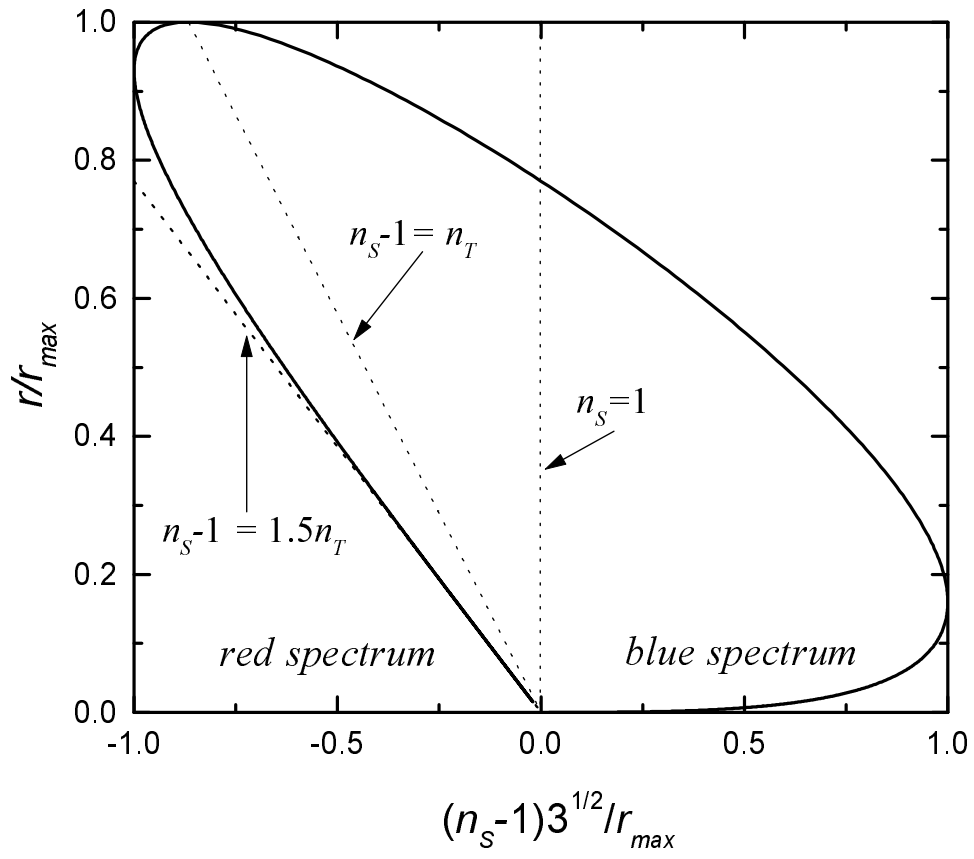


Figure 2: The relationship between  $r$  and  $n_s$  for  $\Lambda$ -inflation ( $r_{max} = \frac{3\sqrt{3}}{2}\epsilon^2$ ,  $r = -2n_T$ ).

Let us now turn to the case where the slow-roll approximation is broken.

For  $\Lambda m$ -inflation eqs.(57) are true except the blue part of the S-spectrum ( $k > k_{cr}$ ) where it must be corrected. Here eqs.(52), (53) are solved explicitly,

$$y < 1 : \quad k^{\frac{3}{2}} \nu_k \simeq \frac{ik\sqrt{\pi x}}{2} H_{\frac{3}{2}p}^{(1)}(x) \xrightarrow{x \ll 1} \frac{caH_0}{\sqrt{2p}} x^{\frac{3}{2}(1-p)},$$

where  $H_p^{(1)}(x)$  is the Hankel function,  $x = k|\eta|$ ,  $c = \frac{p}{\sqrt{2\pi}} \Gamma\left(\frac{3}{2}p\right) 2^{\frac{3}{2}p} = \frac{2^{3p/2}}{3\sqrt{\pi/2}} \Gamma(1 + \frac{3}{2}p)$ . Taking into account the field asymptotics for  $y \ll 1$  (see eq.(44)) we obtain the following S-spectrum in the blue range:

$$k > k_{cr} : \quad q_k \simeq \frac{cH_0}{2\pi\epsilon} \left( \frac{k}{k_{cr}} \right)^{\frac{3}{2}(1-p)}, \quad n_S^{blue} = 4 - 3p > 1. \quad (60)$$

As we see, the spectrum amplitude remains a finite number for  $p \rightarrow 0$  ( $n_S \rightarrow 4$ ).<sup>10</sup> In most applications we usually have  $n_S < 3$  ( $p > \frac{1}{3}$ ); in this case the whole spectrum approximation for the  $\Lambda m$ -inflation looks as follows:

$$q_k = \frac{H_0(1 + y^2)^{\frac{1}{2}}(\tilde{c} + y^2)}{2\pi\epsilon y}, \quad (61)$$

where  $\tilde{c} = \frac{c(1+p)}{p} = \frac{1+p}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}p\right) 2^{\frac{3}{2}(p-1)}$  and  $y$  is taken at horizon crossing (see eq.(41)).

## 6 The T/S effect in $\Lambda$ -inflation

A large T/S  $\sim 1$  (when  $k_{cr} \in (10^{-4}, 10^{-3})$ ) is an intrinsic property of  $\Lambda$ -inflation. Below, we demonstrate it straightforwardly for COBE angular scale (see [19], [20], eq.(1)).

Then the S and T are written as follows:

$$S = \sum_{\ell=2}^{\infty} S_{\ell} \exp \left[ - \left( \frac{2\ell+1}{27} \right)^2 \right], \quad T = \sum_{\ell=2}^{\infty} T_{\ell} \exp \left[ - \left( \frac{2\ell+1}{27} \right)^2 \right], \quad (62)$$

---

<sup>10</sup>This corrects the wrong statement on the divergence of  $q_k$  at  $p \rightarrow 0$  made in some previous publications.

where  $S_\ell$ ,  $T_\ell$  are the corresponding variances in  $\ell$ th harmonic component of  $\Delta T/T$  on the celestial sphere,

$$S_\ell = \sum_{m=-\ell}^{\ell} |a_{\ell m}^{(S)}|^2, \quad T_\ell = \sum_{m=-\ell}^{\ell} |a_{\ell m}^{(T)}|^2, \quad \frac{\Delta T}{T}(\vec{e}) = \sum_{\ell, m, S, T} a_{\ell m}^{(S, T)} Y_{\ell m}(\vec{e}). \quad (63)$$

The calculations can be done for the instantaneous recombination,  $\eta = \eta_E$  [2],

$$\frac{\Delta T}{T}(\vec{e}) = \left( \frac{1}{4} \delta_\gamma - \vec{e} \vec{v}_b + \frac{1}{2} h_{00} \right)_E + \frac{1}{2} \int_E \frac{\partial h_{ik}}{\partial \eta} e^i e^k dx, \quad e^i = (1, -\vec{e}), \quad x \equiv |\vec{x}| = \eta_0 - \eta,$$

where the SW-integral makes a dominant contribution on large scale (see eq.(45)),  $\delta_\gamma$  and  $\vec{v}_b$  are the photon density contrast and baryon peculiar velocity, respectively. The mean  $S_\ell$  and  $T_\ell$  values seen by an arbitrary observer in the matter dominated Universe (e.g. [21]) are explicitly related with the respective power spectra (see eqs.(55)):

$$S_\ell = \frac{2\ell + 1}{25} \int_0^\infty q_k^2 j_\ell^2 \left( \frac{k}{k_0} \right) \frac{dk}{k}, \quad (64)$$

$$T_\ell = \frac{9\pi^2}{16} (2\ell + 1) \frac{(\ell + 2)!}{(\ell - 2)!} \int_0^\infty h_k^2 I_\ell^2 \left( \frac{k}{k_0} \right) \frac{dk}{k}, \quad (65)$$

where  $k_0 = \eta_0^{-1} = \frac{H_0}{2} \simeq 1.6 \times 10^{-4} h \text{ Mpc}^{-1}$ ,

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}(x), \quad I_\ell(x) = \int_0^x \frac{J_{\ell+1/2}(x-y) J_{5/2}(y)}{(x-y)^{5/2} y^{3/2}} dy.$$

We have derived T/S for  $\Lambda m$ -inflation using the approximation (61). The result is presented in Fig.3 as a function of two parameters of the model: the spectrum index in the blue asymptotics  $n_S^{blue}$  (see eq.(60)) and the critical scale  $k_{cr}$  (in units  $k_0$ ). A similar behaviour of T/S is met for  $\Lambda \lambda$ -inflation.

Actually, the two-arm structure of T/S is typical for any  $\Lambda$ -inflation model: T/S gets its maximum at  $k_{cr} \sim k_{COBE} \sim 10^{-3} h \text{ Mpc}^{-1}$  and gradually decays in both, blue ( $k_{cr} < k_{COBE}$ ) and far red ( $k_{cr} \gg k_{COBE}$ ) sectors of the S-mode. To be precise, the T/S-maximum is achieved in the location of  $r$ -maximum (where  $\gamma$  is the largest and thus  $\vartheta = 0$ ). There (and anywhere) the S-spectrum slope is pretty close to HZ for  $\epsilon \ll \epsilon_0$  (cf. eqs.(32), (58)):

$$1 - n_S^{(r_{max})} \simeq -n_T^{(r_{max})} = 2\gamma_{max} \simeq \frac{1}{2} r_{max} \simeq \left( \frac{\epsilon}{\epsilon_0} \right)^2 \ll 1. \quad (66)$$

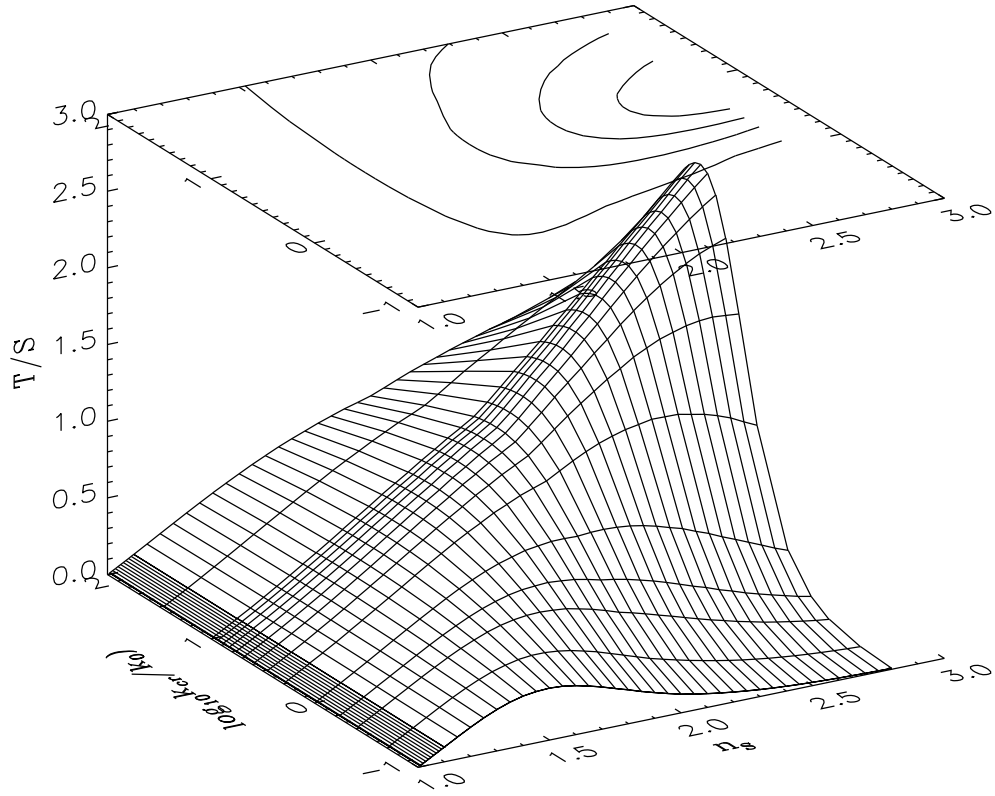


Figure 3:  $T/S$  as a function of  $k_{cr}$  and  $n_S^{blue}$  ( $n_S$  in blue asymptotic) in the case of  $\Lambda m$ -inflation.

It is important that  $T/S$  remains large in a broad  $k$ -region including the point where the S-spectrum is exactly HZ:  $\frac{r_{n_S=1}}{r_{max}} \simeq \frac{4}{9} \left(1 + \frac{\kappa}{2}\right)^{\frac{2}{\kappa}} > \frac{4}{9}$ .

## 7 Discussion

It seems as a paradox that  $T/S$  can be as large as 1 for such a simple model as  $\Lambda$ -inflation. However, it can be easily understood. In fact, the model recalls a case of double inflation where the large  $T/S$  is generated in the intermediate scales between the first and second stages. So, we can assume that it is sufficient to evaluate  $T/S$  by the end of the first stage ( $\varphi \sim \varphi_{cr} \sim 1$ ) where the slow-roll condition is marginally applicable. Here (cf. eqs.(2), (32))

$$\frac{T}{S} \sim \varphi^{-2} \sim 1. \quad (67)$$

Often  $T/S$  is presented as a function of the gravitational-wave-spectrum index  $n_T$  or the inflationary  $\gamma$ -parameter estimated in the given scale, see eq.(2) (e.g. [22], [23], [24], [25], and others). We think this formula is universal for most types of cosmic inflation. We can argue it by plainly noting a clear physical equation,

$$\frac{T}{S} \simeq 3r, \quad (68)$$

where  $r$  is taken in scale where the  $T/S$  is determined ( $k_{COBE} \sim 10^{-3} h \text{ Mpc}^{-1}$ ). The factor  $3^{11}$  takes into account a higher ability of T-mode to contribute to  $\Delta T/T$ . We now see from eqs.(52)-(56) that  $r$  is a number found in the limit  $k|\eta| \ll 1$ :

$$k|\eta| \ll 1 : \quad r = 4\gamma \left| \frac{\nu_k^\lambda}{\nu_k} \right|^2. \quad (69)$$

Implying that the r.h.s. stays frozen outside the horizon ( $k|\eta| < 1$ ) we can estimate  $r$  as the r.h.s. of eq.(69) at inflation horizon crossing time. Thus, we may conclude that

$$r \simeq 4 \left( \gamma \left| \frac{\nu_k^\lambda}{\nu_k} \right|^2 \right)_{k|\eta|=1} \simeq 4\gamma_{k|\eta|=1}. \quad (70)$$

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<sup>11</sup>Or somewhat about 3, to be found more accurately by special investigation elsewhere. Eventually, it is proportional to the ratio of the effective numbers of T and S spin projections on given spherical harmonics, see eqs.(64), (65).



The latter is due to the fact that the functions  $\nu_k^\lambda$  and  $\nu_k$  are close to each other at the horizon crossing: they both start from the same initial conditions (53) and obey the same equations inside the horizon (see eqs.(52))<sup>12</sup>. Notice this argument is more general than the slow-roll-condition validity: actually, according to eqs.(51), (70) the  $r$ -number counts just the difference between the phase space volumes of phonons and gravitons.

So, we see that large T/S is created each time when the  $\gamma$  factor approaches subunity values. It may happen either in the end of inflation (note inflation stops for  $\gamma = 1$ ) or in a numerous intermediate periods during inflationary regime where one type of the inflation is changed for another one. Such a transition periods can be caused by many reasons; e.g. due to a functional change of the dynamical potential in the course of inflation (e.g.  $\Lambda$ -inflation), or a peculiarity in the potential energy shape (e.g. non-analiticity, a step, plateau, or a break of the first or second derivative of  $V(\varphi)$ ), see e.g. [26]), or a change of the inflaton field (e.g. double-inflation), or any type of phase transitions or other evolutionary restructurings of the field Lagrangian that may slower down, terminate, or break up the process of inflation.

Obviously, each particular way of inflation leaves its own imprints in the power spectra and requires special investigation. However, the issue of T/S is a matter of the very generic argument: the inflationary ( $\gamma$ ,  $H$ ) and/or spectral ( $r$ ,  $n_T$ ) parameters estimated in the appropriate energy/scale region. It (the T/S value) is totally independent of the local  $n_S$  and, thus, has nothing to do with a particular S-spectrum shape produced in a given model.

The principal quantity for estimating T/S becomes the energy of inflaton: the Hubble parameter at the inflationary horizon-crossing time,  $H$  [GeV]. The motivation is the following: as the CGW amplitude is always about  $H$  (cf.eq.(57)) and  $q_k \sim 10^{-5}$  (from LSS originated from the adiabatic S-mode), we have

$$\frac{T}{S} \simeq \frac{1}{6} \left( \frac{H}{q_{COBE}} \right)^2 = \left( \frac{H}{6 \times 10^{13} \text{GeV}} \right)^2 \left( \frac{10^{-5}}{q_{COBE}} \right)^2, \quad (71)$$

where  $q_{COBE} \equiv q_{k_{COBE}}$ . So, measuring the T/S brings a vital direct information on the physical energy scale where the cosmic perturbations has been created; a cosmologically noticable T/S could be achieved only if the infla-

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<sup>12</sup> The difference in their evolutions originates only because of various effective potentials  $U^{(\lambda)}$  entering eqs.(52); however both potentials vanish for  $k|\eta| > 1$ .

tion occurred at subPlanckian (GUT) energies,  $H > 10^{13} GeV$ . If the CDPs were generated at smaller energies (e.g. during electroweak transition) then T/S would vanish.

The point we emphasize in this paper is that for  $\Lambda$ -inflation. It brings about two distinguished signatures – a wing-like S-spectrum and the possibility for large T/S – under quite a simple and natural assumption on the potential energy of inflaton: the existence in  $V(\varphi)$  of a *metastable dynamical constant* in addition to an *arbitrary functional*  $\varphi$ -dependent term. It is obviously three independent parameters that determine the degrees of freedom of any  $\Lambda$ -inflation model. They can be, for instance, T/S and the local  $n_S$  (at the COBE scale) as well as  $k_{cr}$  (the scale where S-spectrum is at minimum) or, alternatively, the  $r$ -maximum and its position (the  $k_1$  scale) as well as  $V_0$ . In case if T/S is large, we find quite a definite prediction on the location of the  $\Lambda$ -inflation parameters near GUT energies (see eq.(15)):

$$\frac{T}{S} > 0.1 : \quad \begin{aligned} \sqrt{V_0} &\in \left( \frac{\zeta^{-\frac{\kappa}{2}}}{\sqrt{\kappa/2}}, \zeta \right) \left( \frac{q_{COBE}}{10^{-5}} \right) (7 \times 10^{15} GeV)^2 \\ \frac{\sqrt{\lambda_\kappa/3}}{q_{COBE}} &\in \left( 10^{-\frac{\kappa}{2}}, \frac{\kappa}{2} \zeta \right) (\kappa - 1)^{\frac{1-\kappa}{2}} (2 \times 10^{18} GeV)^{2-\frac{\kappa}{2}} \end{aligned} ,$$

where  $\zeta \equiv 4\epsilon(\kappa - 1)^{\frac{\kappa-1}{\kappa}} \simeq \frac{2(\kappa-1)10^{19} GeV}{\varphi_{cr}} \in (1, 10)$ ; recall these estimates assume only the condition  $T/S > 0.1$  (cf. eqs.(57), (58), (68)).

## 8 Conclusions

Our conclusions are the following:

- \* We introduce a broad class of elementary inflaton models called the  $\Lambda$ -inflation. The inflaton in the local-minimum-point has a *positive residual potential energy*,  $V_0 > 0$ . The hybrid inflation model (at the intermediate evolutionary stage) is a partial case of  $\Lambda$ -inflation; the chaotic inflation is a measure-zero-model in the family of  $\Lambda$ -inflation models.
- \* The S-perturbation spectrum generated in  $\Lambda$ -inflation has a non-power-law *wing-like shape with a broad minimum* where the slope is locally HZ ( $n_S = 1$ ); it is blue,  $n_S > 1$ , (red,  $n_S < 1$ ) on short (large) scales. The T-perturbation spectrum remains always red with the maximum deviation from HZ at the scale near the S-spectrum-minimum.

- \* The cosmic gravitational waves generated in  $\Lambda$ -inflation contribute *maximally* to the SW  $\Delta T/T$ -anisotropy,  $(T/S)_{max} \lesssim 10$ , in scales where the S-spectrum is slightly red or nearly HZ ( $k \lesssim k_{cr}$ ). The T/S remains small ( $\ll 1$ ) in both blue ( $k > k_{cr}$ ) and far red ( $k \ll k_{cr}$ ) S-spectrum asymptotics.
- \* *Three* independent arbitrary parameters determine the fundamental  $\Lambda$ -inflation; they can be the T/S,  $k_{cr}$  (the scale where  $n_S = 1$ ), and  $\sqrt{V_0}$  (the CDP amplitude at  $k_{cr}$  scale; a large value for T/S is expected if  $V_0^{\frac{1}{4}} \sim 10^{16} GeV$ ). This brings high capability in fitting various observational tests to the dark matter cosmologies based on  $\Lambda$ -inflation.

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## APPENDIX: $\Lambda m$ -inflation with $\epsilon^2 < 0.9$

Here, we consider the inflation model with  $\kappa = 2$  and  $p > \frac{2}{3}$  ( $\epsilon^2 < \frac{5}{6}$ ).

Under the latter restriction,  $\vartheta \simeq \text{const} = \frac{3}{2}(1-p)$  during the whole dS stage ( $y < 1$ , cf.eq.(42)) and decays as  $\vartheta \simeq \frac{3f}{4} \simeq -\frac{\epsilon^2}{2y^2}$  for  $y > 1$ . Making use of eqs.(26) we find the following best fit for the whole  $y$ -evolution (analytically exact in the limit  $\epsilon \rightarrow 0$ ):

$$\vartheta \simeq \frac{3}{2} \left( 1 - \sqrt{1-f} \right) = \frac{1.5f}{1 + \sqrt{1-f}}, \quad (A1)$$

$$\sqrt{\frac{\gamma}{2}} \simeq \frac{\epsilon y}{(1+y^2)(1+\sqrt{1-f})}. \quad (A2)$$

where  $f \equiv \frac{2\epsilon^2}{3} \frac{1-y^2}{(1+y^2)^2}$ . The substitution of (A2) into eq.(27) brings about the explicit integration:

$$\epsilon^2 \ln \left( \frac{v\eta}{\sqrt{2}\eta_{cr}} \right) \simeq J(\xi), \quad (A3)$$

where  $\xi \equiv v^2 (1 + \sqrt{1-f}) = v^2 + \sqrt{v^4 + (1-p^2)(v^2-2)}$ ,  $v^2 = 1 + y^2$ ,

$$J(\xi) \equiv \int_1^{\xi} \frac{\xi dy}{y} = \frac{\xi}{2} - 2 + \frac{1}{2} \ln \left[ \left( \frac{\xi-1-p}{3-p} \right)^{1+p} \left( \frac{\xi-1+p}{3+p} \right)^{1-p} \left( \frac{2\xi+1-p^2}{9-p^2} \right)^{\frac{1-p^2}{2}} \right].$$

Obviously, the evolution goes from large  $\xi = 2y^2 \left( 1 + \frac{1+\epsilon^2/6}{y^2} + O\left(\frac{1}{y^4}\right) \right) > 4$  to small  $\xi = (1+p) \left( 1 + \frac{3-p}{2p} y^2 + O(y^4) \right) < 4$ , and  $\xi_{cr} = 4$ . Accordingly, we have the following  $y$ -asymptotics from eq.(A3):

$$y^2 = \begin{cases} \epsilon^2 \ln \left( \frac{\omega_5 \eta}{\eta_{cr}} \right) + (1+p^2) \ln \left( \frac{y_5}{y} \right) + 1, & y > 1, \\ y_6^2 \left( \frac{\omega_6 \eta}{\eta_{cr}} \right)^{3(1-p)}, & y < 1, \end{cases} \quad (A4)$$

where  $\omega_5^{-1} = \sqrt{2} \exp\left(\frac{1}{6}\right)$ ,  $y_5 = \left(\frac{3-p}{2}\right)^{\frac{(1+p)(3-p)}{4(1+p^2)}} \left(\frac{3+p}{2}\right)^{\frac{(1-p)(3+p)}{4(1+p^2)}}$ ,  $\omega_6 = \frac{\eta_{cr}}{\eta_3} = \frac{1}{\sqrt{2}}(3+p)^{\frac{3+p}{6(1+p)}}$ ,  $y_6 = (2p)^{\frac{p}{1+p}}(1+p)^{\frac{p-3}{4}} \exp\left[\frac{3-p}{2(1+p)}\right]$ .

In the allowed region  $p > \frac{2}{3}$ , the coefficients  $\omega_6$  and  $y_6$  remain close to unity. In the slow-roll limit ( $p \rightarrow 1$ ),  $\omega_6 = 2^{\frac{1}{6}}$  and  $y_6 = \sqrt{e}$ .

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